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Enumeration of directed compact site animals in two dimensions

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Abstract. We study the problem of directed compact site animals in two dimensions using exact enumeration methods. The numerical analysis of our animal number series strongly indicates that the critical exponent θ equals zero. We also estimate the non-universal growth parameter $\lambda = 2.661\,85 \pm 0.000\,05$.

The problem of directed animals has received considerable attention recently (Redner and Yang 1982, Dhar *et al* 1982, Nadal *et al* 1982, Day and Lubensky 1982, Redner and Coniglio 1982). This is a modification of the usual isotropic animal problem in which directionality plays a fundamental role. One can define a directed animal as a set of lattice sites, such that each site belonging to the set is reachable from a given fixed site (the origin) by directed paths only. The statistical properties of directed animals have been well studied by field theoretic methods (Day and Lubensky 1982) and by direct enumeration (Redner and Yang 1982, Dhar *et al* 1982).

In this paper, we enumerate directed compact site animals in two dimensions and obtain estimates for the critical exponent θ and inverse critical fugacity or growth parameter λ by analysing our enumeration data.

A directed compact site animal can be defined as a connected cluster of sites having no holes, i.e. all the sites enclosed by the directed animal are occupied. A simply directed animal, however, contains holes in between.

We write the animal generating function as the sum of the weights of all animals, the weight of an animal of size n being x^n:

$$F(x) = \sum_{n=0}^{\infty} F_n x^n \tag{1}$$

where F_n is the number of site animals of size n. For large n, F_n is expected to have the asymptotic form

$$F_n \sim \lambda^n \bar{n}^{\theta}. \tag{2}$$

Here, λ is a lattice dependent growth parameter and θ is a critical exponent.

We enumerate directed compact lattice animals defined on a square lattice by writing the recursion relations between animals with different source sets. These recursions are used to generate a series expansion on a computer. A closed form expression for the recursion relations used in the cluster enumeration can be written as (see the appendix)

$$F(x, y) = -xF^1(x) + \frac{y}{1-y} + \frac{F(x, xy)}{y(1-y)^2}. \tag{3}$$

We have generated animals up to size 19 on a Cyber 170-730 computer (see the appendix).

We now consider the analysis of the animal number series to obtain estimates for θ and λ. We form the estimates $\theta_n = \ln(F_n^2/F_{n-1}F_{n+1})/\ln(1-1/n^2)$ by using equation (2) and three successive terms of the series expansion. For $n \to \infty$, one can show that $\theta_n \to \theta$. Our results strongly indicate that $\theta_n \to 0$ as the number of sites n become large. Thus we find that the critical exponent $\theta = 0$ for two-dimensional directed compact animals. The growth parameter λ is obtained as the $n \to \infty$ limit of λ_n, where $\lambda_n = F_n/F_{n-1}$ is the ratio of successive generating function coefficients. Following Gaunt *et al* (1976), we obtain estimates for λ based on linearly extrapolating λ_n against $1/n$, i.e. $n\lambda_n - (n-1)\lambda_{n-1}$. Our results for θ_n and λ_n are displayed in table 1.

We have also used an alternative method following Zinn-Justin (1981) to estimate θ and λ. This method involves calculation of θ_n and λ_n from three second successive terms of the series expansion. We calculate θ_n by using the relation:

$$\theta_n = \frac{-2(a_n + a_{n-2})}{(a_n - a_{n-2})^2}$$

where a_n is defined as

$$a_n = -\left(\frac{F_n F_{n-4}}{F_{n-2}}\right)^{-1}$$

and θ_n corresponds to $1 - \gamma_n$ in Zinn-Justin's notation. The value of λ_n is estimated by

$$\lambda_n = \left(\frac{F_n}{F_{n-4}}\right)^{1/4} \exp\left(-\frac{(a_n + a_{n-2})}{2a_n(a_n - a_{n-2})}\right).$$

Table 1. Estimates for θ_n calculated from three successive terms of the series expansion and λ_n as the ratios of successive generating function coefficients. Extrapolated values of λ_n are also shown.

n	θ_n	λ_n	Extrapolated values of λ_n†
5	0.078 50	2.647 058	
6	0.018 01	2.655 555	2.698 04
7	0.053 61	2.656 903	2.664 99
8	0.021 56	2.659 842	2.680 41
9	0.017 55	2.660 746	2.667 97
10	0.007 93	2.661 326	2.666 54
11	0.006 97	2.661 538	2.663 65
12	0.004 30	2.661 692	2.663 38
13	0.002 36	2.661 772	2.662 73
14	0.001 74	2.661 809	2.662 29
15	0.000 87	2.661 833	2.552 16
16	0.000 64	2.661 843	2.661 99
17	0.000 36	2.661 850	2.661 96
18	0.000 22	2.661 853	2.661 90
19	—	2.661 855	2.661 89

† Best extrapolated value $\lambda = 2.6619 \pm 0.000\,05$.

We find strong evidence in support of our earlier observation that the exponent $\theta = 0$. The value of λ is estimated to be $2.661\,85 \pm 0.000\,05$ which is not much different from our earlier result. This approach minimises any even–odd effect and the results shown in table 2 seem to be slightly better behaved.

In summary, we have studied some properties of directed compact animals in two dimensions. We observe that the exponent $\theta = 0$, which is different from $\theta = \frac{1}{2}$ for directed animals and $\theta = 1$ for undirected animals in two dimensions.

Table 2. Estimates for θ_n and λ_n calculated from three second successive terms of the series expansion.

n	θ_n	λ_n†
9	0.017 537	2.664 62
10	0.010 098	2.663 89
11	0.014 159	2.665 58
12	0.004 220	2.662 88
13	0.001 731	2.662 22
14	0.001 051	2.662 09
15	0.000 708	2.662 03
16	0.000 351	2.661 94
17	0.000 171	2.661 89
18	0.000 097	2.661 88
19	0.000 051	2.661 86

† Best extrapolated value $\lambda = 2.661\,85 \pm 0.000\,05$.

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Appendix

We write the animal generating function as

$$F(x) = \sum_{n=0}^{\infty} F_n x^n \tag{A1}$$

where F_n is the number of site animals of size n. The generating function exhibits power-law singularity of the form

$$F(x) \sim |x_c - x|^{\theta - 1} \tag{A2}$$

as $x \to x_c$. The exponent θ is a critical exponent. We also define the exponent θ directly in terms of cluster size n as

$$F_n \sim n^{-\theta} \lambda^n \qquad \text{(for large } n\text{)}. \tag{A3}$$

From equations (A1) and (A3) it is clear that the singular behaviour of the generating function is recovered in the limit $n \to \infty$. This implies that the critical exponent θ characterising the singularity of the generating function is obtained only in the large n limit.

To understand the formation of recursion relations, let us consider a $3 \times \infty$ strip, i.e. a lattice of width 3 and length ∞. It is easy to write down relations between the animals with different source sets. A source set can be defined as a set of points such that each site in the animal is reachable from at least one of the points in the source using directed paths. A source set is a subset of the animal.

We write down the recursion relations between animals with source sites at column 0 and at column 1 of a $3 \times \infty$ strip. One such relation between the animals with all the three sites occupied at column 0 and one or more sites occupied at column 1 is written as

$$F_n^{111} = F_{n-3}^{111} + F_{n-2}^{110} + F_{n-2}^{101} + F_{n-2}^{011} + F_{n-1}^{100} + F_{n-1}^{010} + F_{n-1}^{001}. \qquad \text{(A4)}$$

Superscripts in equation (A4) represent a given set of occupied sites in the source. The number of occupied sites and their position in column 1 generates different configurations and each site in these configurations is connected to at least one site in column 0. Using equation (A1) in (A4) we obtain

$$\begin{aligned} F^{111}(x) &= x^3 F^{111}(x) + x^2(F^{110}(x) + F^{101}(x) + F^{011}(x)) \\ &\quad + x(F^{100}(x) + F^{010}(x) + F^{001}(x)) + 1. \end{aligned} \qquad \text{(A5)}$$

The choice of periodic boundary conditions further simplifies equation (A4) and we obtain

$$F^{111}(x) = x^3 F^{111}(x) + 3x^2 F^{110}(x) + 3x F^{100}(x) + 1. \qquad \text{(A6)}$$

Similar recursions can be written for $F^{110}(x)$ and $F^{100}(x)$. In this way one can write down recursion relations for non-compact animals on a lattice of any finite width.

In the problem of directed compact animals we consider the cluster extending from the lower left-hand corner of a square lattice. Therefore we shall be concerned with those source sets where all the source sites lie on the line $x + y = \text{constant}$; (x, y) are the coordinates of the source sites.

Since all the sites enclosed by a compact animal are occupied, we can easily write down the recursion relations as

$$\begin{aligned} F^1(x) &= 1 + 2xF^1(x) + x^2 F^{11}(x) \\ F^{11}(x) &= 1 + 3xF^1(x) + 2x^2 F^{11}(x) + X^3 F^{111}(x) \\ F^{111}(x) &= 1 + 4xF^1(x) + 3x^2 F^{11}(x) + 2x^3 F^{111}(x) + x^4 F^{1111}(x) \end{aligned} \qquad \text{(A7)}$$

and so on.

The set of these coupled recursions can be written in a closed form as

$$F(x, y) = -xF^1(x) + \frac{y}{(1-y)} + \frac{F(x, xy)}{y(1-y)^2} \qquad \text{(A8)}$$

where

$$F(x, y) = \sum_{L=1}^{\infty} y^L F^L(x)$$

$$F(x, xy) = \sum_{L=1}^{\alpha} (xy)^L F^L(x). \qquad \text{(A9)}$$

We have obtained equation (A8) by multiplying the first, second, ..., recursions of the set (A6) by $y, y^2, \dots$, respectively, and summing over y. Comparison of coefficients of a Taylor series expansion of $F(x, y)$ around $y=0$ with those of $F(x, y)$ in equation (A9) yields

$$F^1(x) = \frac{\partial F(x, y)}{\partial y}\bigg|\, y=0$$

$$F^{11}(x) = 1/2!\, \frac{\partial F^2(x, y)}{\partial y^2}\bigg|\, y=0 \tag{A10}$$

and so on.

We can generate all the recursion relations of the set (A7) from (A10). Using these recursion relations we have generated a series expansion for directed compact animals in two dimensions. The series expansion is

$$\begin{aligned} F^1(x) = 1 &+ 2x + 5x^2 + 13x^3 + 34x^4 + 90x^5 + 239x^6 + 635x^7 + 1689x^8 + 4494x^9 \\ &+ 11\,960x^{10} + 31\,832x^{11} + 84\,727x^{12} + 225\,524x^{13} + 600\,302x^{14} \\ &+ 1597\,904x^{15} + 4253\,371x^{16} + 11\,321\,838x^{17} \\ &+ 30\,137\,079x^{18} + 80\,220\,557x^{19} + O(x^{20}). \end{aligned}$$

(The series expansion is obtained by substituting, say, $F^{111}(x)$ from the third equation of (A7) into the second equation and continuing the procedure till we get an expansion for $F^1(X)$.) $F^1(x)$ is the same as $F(x)$ defined earlier. The superscript 1 denotes that the animals are generated from a single seed.

References

Day A R and Lubensky T C 1982 *J. Phys. A: Math. Gen.* **15** L285
Dhar D, Phani M K and Barma M 1982 *J. Phys. A: Math. Gen.* **15** L279
Gaunt D S, Sykes M F and Ruskin H 1976 *J. Phys. A: Math. Gen.* **9** 1899
Nadal J P, Derrida B and Vannimenus J 1982 *J. Physique* **43** 1561
Redner S and Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** L273
Redner S and Yang Z R 1982 *J. Phys. A: Math. Gen.* **15** L177
Zinn-Justin J 1981 *J. Physique* **42** 783